Barker arrays II — odd number of elements

Jonathan Jedwab

Sheelagh Lloyd

Miranda Mowbray

Hewlett-Packard Laboratories, Filton Road,

Stoke Gifford, Bristol BS12 6QZ, U.K.

Abstract

A Barker array is a two-dimensional array with elements ± 1 such that all out-of-phase aperiodic autocorrelation coefficients are 0, 1 or -1. No $s \times t$ Barker array with $s, t > 1$ and $(s, t) \neq (2, 2)$ is known and it is conjectured that none exists. Nonexistence results for a class of arrays that includes Barker arrays have been previously given, in the case st even. We prove nonexistence results for this class of arrays in the case st odd, providing further support for the Barker array conjecture.

Keywords Barker array, aperiodic autocorrelation, binary array, nonexistence.

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Abbreviated title Barker Arrays II

1 Introduction

In a previous paper [2] we defined binary arrays with *Barker structure*, a class that contains all $s \times t$ Barker arrays with $st > 2$, and proved restrictions on s, t for the case st even. In this paper we present nonexistence results for the case st odd, providing further support for Alquaddoomi and Scholtz's conjecture [1].

We shall use the notation of [2].

2 Row and column sum equations

We first obtain equations in the row and column sums of an $s \times t$ binary array with Barker structure, where s, t are odd. Using Lemma 1 and Definition 1 (*iii*) of [2], we obtain:

Lemma 1 *Let A be an $s \times t$ binary array with Barker structure where s, t are odd. Let (x_i) and (y_j) be the row and column sums of A. Then each x_i and y_j is an odd integer, and*

$$\sum_i x_i x_{i+u} = \begin{cases} kt & \text{for all } u \text{ even and } u \neq 0 \\ 0 & \text{for all } u \text{ odd} \\ st + k(t-1) & \text{for } u = 0, \end{cases} \tag{1}$$

$$\sum_j y_j y_{j+v} = \begin{cases} ks & \text{for all } v \text{ even and } v \neq 0 \\ 0 & \text{for all } v \text{ odd} \\ st + k(s-1) & \text{for } v = 0, \end{cases} \tag{2}$$

where $k = 1$ or -1 and $k \equiv st \pmod 4$.

We derive all our results from an analysis of equations (1) and (2), although we do not find a general solution. Throughout, we consider solutions only to (1), combining conditions on s and t obtained from both equations at the end.

We can deduce from Lemma 1 an expression for the *imbalance* $\sum_i \sum_j a_{ij} \equiv \sum_i x_i$ of the array A.

Lemma 2 *Let $s, t, (x_i : 0 \le i < s)$ be integers satisfying* (1)*, where $k = 1$ or -1 and $k \equiv st \pmod 4$. Then*

$$\left(\sum_i x_i\right)^2 = \begin{cases} 2st - 1 & \text{for } st \equiv 1 \pmod 4 \\ 1 & \text{for } st \equiv 3 \pmod 4 \end{cases}$$

Proof

$$\begin{aligned} \left(\sum_i x_i\right)^2 &= \sum_i x_i^2 + 2\sum_i \sum_{j>i} x_i x_j \\ &= \sum_i x_i^2 + 2\sum_i \sum_{u>0} x_i x_{i+u}, \end{aligned}$$

putting $j = i + u$. Therefore

$$\begin{aligned} \left(\sum_i x_i\right)^2 &= \sum_i x_i^2 + 2\sum_{u=1}^{s-1}\left(\sum_i x_i x_{i+u}\right) \\ &= st + k(t-1) + 2kt(s-1)/2, \end{aligned}$$

on substitution from (1). Hence

$$\begin{aligned} \left(\sum_i x_i\right)^2 &= (k+1)st - k \\ &= \begin{cases} 2st - 1 & \text{for } st \equiv 1 \pmod 4 \\ 1 & \text{for } st \equiv 3 \pmod 4, \end{cases} \end{aligned}$$

using the given value for k. □

A consequence of Lemma 2 is that $2st - 1$ is a square when $st \equiv 1 \pmod 4$, as noted in Theorem 2 (*ii*) of [2].

In the case $t = 1$, the possible values of s are determined by known results on Barker sequences.

Theorem 1 *Let $s > 1$ be an odd integer and let $t = 1$. Then there exists an $s \times t$ binary array with Barker structure if and only if $s = 3, 5, 7, 11$ or 13.*

Proof Let A be an $s \times t$ binary array with Barker structure. Let (x_i) be the row sums of A. Since $t = 1$, (x_i) is a binary sequence and from (1),

$$\sum_i x_i x_{i+u} = \begin{cases} k & \text{for all } u \text{ even and } u \neq 0 \\ 0 & \text{for all } u \text{ odd} \\ s & \text{for } u = 0, \end{cases} \tag{3}$$

where $k = 1$ or -1. Therefore (x_i) is a Barker sequence of odd length $s > 1$, and so [3] $s = 3$, 5, 7, 11 or 13.

The converse is implied by the existence of a Barker sequence with each of these lengths. □

We henceforth consider $s, t > 1$. Our results are all based on the observation that any prime dividing t divides exactly $s - 1$ of the (x_i).

Lemma 3 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying* (1)*, where $s \geq 2$ and $k = 1$ or -1. Let p be a prime dividing t. Then there exists a unique integer $0 \leq I < s$ such that*

(i) $p \mid x_i$ *if and only if* $i \neq I$

(ii) $x_I^2 \equiv -k \pmod{p}$.

Proof Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying (1). Since

$$p \mid t, \tag{4}$$

equations (1) show that

$$p \mid \sum_i x_i x_{i+u} \text{ for all } 0 < u < s.$$

By Lemma 5 of [2], for some $0 \leq I < s$,

$$p \mid x_i \text{ for all } i \neq I. \tag{5}$$

Put $u = 0$ in (1),

$$\begin{aligned} \sum_i x_i^2 &= st + k(t-1) \\ &\equiv -k \pmod{p}, \end{aligned}$$

from (4). Then from (5),

$$x_I^2 \equiv -k \pmod{p}.$$

This shows that $p \nmid x_I$, because $k = 1$ or -1. Combining with (5),

$$p \,|\, x_i \text{ if and only if } i \neq I.$$

Given p and the (x_i), it is clear that I is unique. □

Corollary 1 *Let A be an $s \times t$ binary array with Barker structure where s, t are odd, $s > 1$ and $st \equiv 1 \pmod{4}$. Then $s \equiv t \equiv 1 \pmod{4}$ and each prime p dividing t satisfies $p \equiv 1 \pmod{4}$.*

Proof Let (x_i) be the row sums of A. From Lemma 1, the (x_i) satisfy equations (1), where $k = 1$. Let p be a prime dividing t. Then from Lemma 3 (ii),

$$x_I^2 \equiv -1 \pmod{p}$$

for some $0 \leq I < s$. Now p is odd, since $p \,|\, t$, and so

$$p \equiv 1 \pmod{4}. \tag{6}$$

Since (6) holds for any prime p dividing t, we have $t \equiv 1 \pmod{4}$. Then from $st \equiv 1 \pmod{4}$ we also have $s \equiv 1 \pmod{4}$. □

For a given prime p dividing t, the value of I is uniquely determined by the (x_i). In some cases the values of only p, s and t are sufficient to determine or restrict the value of I. This leads to restrictions on s and t, and is the objective of our analysis.

We first show that $I \neq 0, s-1$ for any prime p.

Lemma 4 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying (1), where $s > 1$ is odd, $x_i \neq 0$ for at least one odd i, and $k = 1$ or -1. Let p be a prime dividing t and let $0 \leq I < s$ be the unique integer such that $p \,|\, x_i$ if and only if $i \neq I$. Then $I \neq 0$, $s - 1$.*

Proof The existence of I is given by Lemma 3 (i). Suppose, if possible, that $I = 0$ or $s-1$. By symmetry we may relabel the (x_i), if necessary, so that $I = s-1$ and

$$p \,|\, x_i \text{ if and only if } i \neq s-1. \tag{7}$$

Since $x_i \neq 0$ for at least one odd i, we may define r to be the largest integer for which

$$p^r \,|\, x_{2j-1} \text{ for all } 1 \leq j \leq (s-1)/2. \tag{8}$$

From (7), $r \geq 1$. Now for any $1 \leq j \leq (s-1)/2$, put $u = s-2j$ in (1) to obtain

$$\sum_{i=0}^{2j-2} x_i x_{i+s-2j} + x_{2j-1} x_{s-1} = 0. \tag{9}$$

Since s is odd, exactly one of i, $i+s-2j$ is even and the other is odd, for all i. Furthermore from (7),

$$p \,|\, x_i \text{ for all even } i \neq s-1$$

while from (8),

$$p^r \,|\, x_i \text{ for all odd } i.$$

Therefore $p^{r+1} \,|\, \sum_{i=0}^{2j-2} x_i x_{i+s-2j}$ and then from (9),

$$p^{r+1} \,|\, x_{2j-1} x_{s-1}.$$

Now p is prime and by (7), $p \not| \, x_{s-1}$, so we conclude that

$$p^{r+1} \,|\, x_{2j-1} \text{ for all } 1 \leq j \leq (s-1)/2.$$

This contradicts the maximality of r. □

We next fix the parity of I.

Lemma 5 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying* (1)*, where $s > 1$ is odd, x_i is odd for all i, and $k = 1$ or -1. Let p be a prime dividing t and let $0 \leq I < s$ be the unique integer such that $p \,|\, x_i$ if and only if $i \neq I$. Then $I \equiv (s-1)/2 \pmod 2$.*

Proof Summing equations (1) over all odd values of u,

$$\sum_{v\geq 0}\sum_i x_i x_{i+2v+1} = 0.$$

Straightforward manipulation leads to

$$\sum_i x_{2i}\sum_j x_{2j+1} = 0.$$

Therefore either $\sum_i x_{2i} = 0$ or $\sum_j x_{2j+1} = 0$.

Suppose firstly that $\sum_i x_{2i} = 0$. Then I is odd, since $p \mid x_{2i}$ for all $2i \neq I$. Also $\sum_i x_{2i}$ is the sum of exactly $(s+1)/2$ non-zero terms, each of which by hypothesis is odd, and so $(s+1)/2 \equiv 0 \pmod 2$. Therefore

$$I \text{ is odd and } (s+1)/2 \equiv 0 \pmod 2. \tag{10}$$

If instead we suppose that $\sum_j x_{2j+1} = 0$ then, by similar reasoning,

$$I \text{ is even and } (s-1)/2 \equiv 0 \pmod 2. \tag{11}$$

We combine (10) and (11) as

$$I \equiv (s-1)/2 \pmod 2.$$

□

We now prove two lemmas constraining the (x_i), given the value of I.

Lemma 6 *Let $s, (x_i : 0 \leq i < s)$ be integers and let p be a prime such that $p^2 \mid \sum_i x_i x_{i+u}$ for all $0 < u < s$. Let $0 \leq I < s/2$ be an integer such that $p \mid x_i$ if and only if $i \neq I$. Then $p^2 \mid x_j$ for all $2I < j < s$.*

Proof Let j satisfy

$$2I < j < s. \tag{12}$$

Put $u = j - I$ so that

$$p^2 \mid \sum_i x_i x_{i+j-I}. \tag{13}$$

Now

$$p \mid x_i \text{ for all } i \neq I \tag{14}$$

and so p^2 divides each product $x_i x_{i+j-I}$ in (13) unless $i = I$ or $i + j - I = I$. But from (12), $i + j - I > I$ and so p^2 divides each product $x_i x_{i+j-I}$ in (13) except $x_I x_j$. Therefore

$$p^2 \mid x_I x_j.$$

But $p \nmid x_I$ by (14), and so $p^2 \mid x_j$. □

Lemma 7 *Let $s, (x_i : 0 \leq i < s)$ be integers and let p be a prime such that $p^2 \mid \sum_i x_i x_{i+u}$ for all $0 < u < s$. Let $0 \leq I < s$ be an integer such that $p \mid x_i$ if and only if $i \neq I$.*

- *(i) Suppose that $p \,||\, x_j$ for some $0 \leq j < s$. Then $0 \leq 2I - j < s$ and $p \,||\, x_{2I-j}$.*
- *(ii) Let j satisfy $0 \leq j < s$ and $0 \leq 2I - j < s$. Then $p^2 \mid x_j$ if and only if $p^2 \mid x_{2I-j}$.*

Proof

(i) Let $p \,||\, x_j$ for some $0 \leq j < s$. By a similar argument to that used in the proof of Lemma 6, to avoid the false conclusion $p^2 \mid x_j$ we require that $i + j - I = I$ has a solution for some $0 \leq i < s$. Consequently $0 \leq 2I - j < s$ and

$$p^2 \mid x_j + x_{2I-j}.$$

Then $p \,||\, x_j$ if and only if $p \,||\, x_{2I-j}$.

(ii) Let j satisfy $0 \leq j < s$ and $0 \leq 2I - j < s$. Then similar reasoning shows that

$$p^2 \mid x_j + x_{2I-j},$$

from which $p^2 \mid x_j$ if and only if $p^2 \mid x_{2I-j}$.

□

The equation $x_0x_{s-1} = \pm t$, obtained by putting $u = s-1$ in (1), is of particular importance. Given a prime p dividing t, we shall often be able to obtain information about the (x_i) from the distribution of powers of p between x_0 and x_{s-1}.

Definition Let p be a prime and x, y be integers where $x \geq 0$. Let $p^x \,|\, y$ and $p^{x+1} \not| \, y$. Then p^x is said to *strictly divide* y, written $p^x \,||\, y$.

Lemma 8 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying* (1)*, where $s > 1$ is odd, $x_i \neq 0$ for at least one odd i, and $k = 1$ or -1. Let p be a prime such that $p^\alpha \,||\, t$ for some integer $\alpha \geq 1$. Then $\alpha \geq 2$ and $p^\gamma \,||\, x_0$, $p^{\alpha-\gamma} \,||\, x_{s-1}$ for some $0 < \gamma < \alpha$.*

Proof Put $u = s-1$ in (1),

$$x_0x_{s-1} = \pm t. \tag{15}$$

Since $p^\alpha \,||\, t$, we then have $p^\gamma \,||\, x_0$, $p^{\alpha-\gamma} \,||\, x_{s-1}$ for some $0 \leq \gamma \leq \alpha$. By Lemma 4, $p \,|\, x_0, x_{s-1}$. Therefore $0 < \gamma < \alpha$ and, from (15), $p^2 \,|\, t$. □

Corollary 2 *Let A be an $s \times t$ binary array with Barker structure where s, t are odd and $s > 1$. Then each prime p dividing t satisfies $p^2 \,|\, t$.*

Proof Let (x_i) be the row sums of A. From Lemma 1, the $(x_i : 0 \leq i < s)$ are odd integers satisfying equations (1), where $k = 1$ or -1. Let p be a prime dividing t. Then $p^2 \,|\, t$ by Lemma 8. □

3 The case $\gamma = 1$

In this section we consider solutions to equations (1) for which $p \,||\, x_0$ and $p^{\alpha-1} \,||\, x_{s-1}$, where p is a prime. The value of I is then determined by s and α, which in turn gives restrictions on s in terms of α.

Lemma 9 *Let $\alpha \geq 2$ and $s, (x_i : 0 \leq i < s)$ be integers and let p be a prime such that*

$$p^\alpha \quad | \quad \sum_i x_i x_{i+u} \text{ for all } 0 < u < s, \tag{16}$$

$$p \quad || \quad x_0, \tag{17}$$

$$p^{\alpha-1} \quad || \quad x_{s-1}. \tag{18}$$

Let $0 \leq I < s$ be an integer such that

$$p \,|\, x_i \text{ if and only if } i \neq I. \tag{19}$$

If $\alpha = 2$ then $I = (s-1)/2$. If $\alpha > 2$ then for all $1 \leq \beta \leq \alpha - 2$,

$$(\beta+1)I \quad < \quad s-1, \tag{20}$$

$$p^{\alpha-\beta} \quad | \quad x_{s-1-j} \text{ for all } 0 \leq j < \beta I, \tag{21}$$

$$p^{\alpha-\beta-1} \quad || \quad x_{s-1-\beta I}. \tag{22}$$

Proof Since $\alpha \geq 2$, apply Lemma 7 (i) with $j = 0$ to give

$$2I \quad < \quad s, \tag{23}$$

$$p \quad || \quad x_{2I}. \tag{24}$$

We show, by induction on j, that

$$p^{\alpha-1} \,|\, x_{s-1-j} \text{ for all } 0 \leq j < I. \tag{25}$$

The case $j = 0$ is given by (18). Assume that for some

$$1 \leq j < I, \tag{26}$$

$$p^{\alpha-1} \,|\, x_{s-1-k} \text{ for all } 0 \leq k < j. \tag{27}$$

Put $u = s-1-j$ in (16),

$$p^\alpha \,|\, (x_0 x_{s-1-j} + \sum_{i=1}^{j} x_i x_{i+s-1-j}). \tag{28}$$

Now by (26), $j < I$ and so by (19), $p \,|\, x_i$ for all $1 \le i \le j$. Furthermore by (27), $p^{\alpha-1} \,|\, x_{i+s-1-j}$ for all $1 \le i \le j$. Therefore $p^\alpha \mid \sum_{i=1}^{j} x_i x_{i+s-1-j}$ and so by (28),

$$p^\alpha \,|\, x_0 x_{s-1-j}.$$

Using (17) we conclude that $p^{\alpha-1} \,|\, x_{s-1-j}$, completing the induction on j and proving (25).

Put $u = s - 1 - I$ in (16),

$$p^\alpha \,|\, (x_0 x_{s-1-I} + \sum_{i=1}^{I-1} x_i x_{i+s-1-I} + x_I x_{s-1}). \tag{29}$$

From (19) and (25), $p^\alpha \mid \sum_{i=1}^{I-1} x_i x_{i+s-1-I}$. Therefore from (29),

$$p^\alpha \,|\, (x_0 x_{s-1-I} + x_I x_{s-1}). \tag{30}$$

From (19), $p \not| \, x_I$ and so by (18), $p^{\alpha-1} \,||\, x_I x_{s-1}$. Therefore from (30),

$$p^{\alpha-1} \,||\, x_0 x_{s-1-I}. \tag{31}$$

In the case $\alpha = 2$ we conclude from (17) and (31) that $p \not| \, x_{s-1-I}$ and then from (19), $s-1-I = I$ or equivalently $I = (s-1)/2$, as required. For the rest of the proof take $\alpha > 2$. Then (17) and (31) imply that

$$p^{\alpha-2} \,||\, x_{s-1-I}, \tag{32}$$

and, since $\alpha > 2$ and $p \not| \, x_I$, we deduce $s - 1 - I \ne I$. Combine this with (23) to give

$$2I < s - 1. \tag{33}$$

We now prove (20)—(22) for all $1 \le \beta \le \alpha - 2$ by induction on β. The case $\beta = 1$ is given by (33), (25) and (32) respectively. Assume that for some

$$2 \le \beta \le \alpha - 2, \tag{34}$$

(20)—(22) hold for $\beta - 1$, so that

$$\beta I \quad < \quad s - 1, \tag{35}$$

$$p^{\alpha-\beta+1} \quad | \quad x_{s-1-j} \text{ for all } 0 \leq j < (\beta-1)I, \tag{36}$$

$$p^{\alpha-\beta} \quad || \quad x_{s-1-(\beta-1)I}. \tag{37}$$

Then to complete the induction on β we must prove the following:

$$(\beta+1)I \quad < \quad s-1, \tag{38}$$

$$p^{\alpha-\beta} \quad | \quad x_{s-1-j} \text{ for all } 0 \leq j < \beta I, \tag{39}$$

$$p^{\alpha-\beta-1} \quad || \quad x_{s-1-\beta I}. \tag{40}$$

We first prove (38). From (36) and (37),

$$p^{\alpha-\beta} \,|\, x_{s-1-j} \text{ for all } 0 \leq j \leq (\beta-1)I. \tag{41}$$

By (34), $\alpha - \beta \geq 2$ and so from (41),

$$p^2 \,|\, x_{s-1-j} \text{ for all } 0 \leq j \leq (\beta-1)I.$$

Comparison with (24) shows that

$$2I < s - 1 - (\beta-1)I,$$

which is equivalent to (38).

We next prove (39). From (36), it is sufficient to establish

$$p^{\alpha-\beta} \,|\, x_{s-1-j} \text{ for all } (\beta-1)I \leq j < \beta I, \tag{42}$$

which we prove by induction on j. The case $j = (\beta-1)I$ is given by (37). Assume that for some

$$(\beta-1)I + 1 \leq j < \beta I, \tag{43}$$

$$p^{\alpha-\beta} \,|\, x_{s-1-k} \text{ for all } (\beta-1)I \leq k < j. \tag{44}$$

Put $u = s - 1 - j$ in (16),

$$p^{\alpha} \,|\, \sum_i x_i x_{i+s-1-j}. \tag{45}$$

By (34), $\beta \geq 2$ and so

$$\alpha - \beta + 1 \leq \alpha - 1. \tag{46}$$

Therefore from (45),

$$p^{\alpha-\beta+1} \mid \sum_i x_i x_{i+s-1-j}. \tag{47}$$

Now by (43), $j \geq (\beta - 1)I + 1$ and by (34), $\beta \geq 2$, so

$$j \geq I + 1. \tag{48}$$

We can therefore write (47) in the form

$$p^{\alpha-\beta+1} \mid (x_0 x_{s-1-j} + \sum_{1 \leq i < I, I < i \leq j} x_i x_{i+s-1-j} + x_I x_{I+s-1-j}). \tag{49}$$

By (41) and (44),

$$p^{\alpha-\beta} \mid x_{i+s-1-j} \text{ for all } 1 \leq i \leq j.$$

Together with (19), this implies

$$p^{\alpha-\beta+1} \mid \sum_{1 \leq i < I, I < i \leq j} x_i x_{i+s-1-j}$$

and so from (49),

$$p^{\alpha-\beta+1} \mid (x_0 x_{s-1-j} + x_I x_{I+s-1-j}). \tag{50}$$

By (48), $j \geq I + 1$ and by (43), $j < \beta I$ and so by (36), $p^{\alpha-\beta+1} \mid x_{I+s-1-j}$. Therefore from (50),

$$p^{\alpha-\beta+1} \mid x_0 x_{s-1-j}.$$

From (17) we conclude that

$$p^{\alpha-\beta} \mid x_{s-1-j},$$

completing the induction on j and proving (42), and hence (39).

We lastly prove (40). Put $u = s - 1 - \beta I$ in (16) and use (46) to show that

$$p^{\alpha-\beta+1} \mid (x_0 x_{s-1-\beta I} + \sum_{1 \leq i < I, I < i \leq \beta I} x_i x_{i+s-1-\beta I} + x_I x_{s-1-(\beta-1)I}). \tag{51}$$

By (39), $p^{\alpha-\beta} \,|\, x_{i+s-1-\beta I}$ for all $1 \leq i \leq \beta I$. Together with (19), this implies

$$p^{\alpha-\beta+1} \,|\, \sum_{1 \leq i < I, I < i \leq \beta I} x_i x_{i+s-1-\beta I},$$

and so from (51),

$$p^{\alpha-\beta+1} \,|\, (x_0 x_{s-1-\beta I} + x_I x_{s-1-(\beta-1)I}). \tag{52}$$

From (19), $p \not| \, x_I$ and so by (37), $p^{\alpha-\beta} \,||\, x_I x_{s-1-(\beta-1)I}$. Therefore from (52),

$$p^{\alpha-\beta} \,||\, x_0 x_{s-1-\beta I}.$$

We conclude from (17) that

$$p^{\alpha-\beta-1} \,||\, x_{s-1-\beta I},$$

which is (40).

This completes the induction on β, proving (20)—(22) for all $1 \leq \beta \leq \alpha - 2$. □

We now use Lemma 9 to prove the intended result of this section.

Theorem 2 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying (1), where $s > 1$ is odd and $k = 1$ or -1. Let p be a prime such that $p^\alpha \,||\, t$ for some integer $\alpha \geq 2$, and $p \,||\, x_0$. Then*

(i) $s \equiv 1 \pmod{\alpha}$

(ii) if x_i is odd for all i then $(s-1)(\alpha-2) \equiv 0 \pmod{4\alpha}$

(iii) $I = (s-1)/\alpha$ is the unique integer such that $p \,|\, x_i$ if and only if $i \neq I$

(iv) for all $2 \leq r \leq \alpha$,

$$p^r \quad | \quad x_j \text{ for all } j > rI, \tag{53}$$

$$p^{r-1} \quad || \quad x_{rI}. \tag{54}$$

Proof By Lemma 3 (i), let I be the unique integer such that $p \,|\, x_i$ if and only if $i \neq I$. Take $u = s - 1$ in (1) to give $x_0 x_{s-1} = \pm t$. Then $p^\alpha \,||\, t$ and $p \,||\, x_0$ imply

$$p^{\alpha-1} \,||\, x_{s-1}, \tag{55}$$

and we may apply Lemma 9.

We first prove that

$$I = (s-1)/\alpha. \tag{56}$$

If $\alpha = 2$ then (56) is given directly by Lemma 9. Suppose that $\alpha > 2$. Apply Lemma 9, taking $\beta = 1$ in (20) to give

$$2I < s - 1 \tag{57}$$

and taking $\beta = \alpha - 2$ in (21) and (22) to give

$$p^2 \quad | \quad x_{s-1-j} \text{ for all } 0 \leq j < (\alpha-2)I, \tag{58}$$

$$p \quad || \quad x_{s-1-(\alpha-2)I}. \tag{59}$$

From (57) and Lemma 6,

$$p^2 \,|\, x_j \text{ for all } 2I < j < s. \tag{60}$$

Put $j = 0$ in Lemma 7 (i) to show

$$p \,||\, x_{2I}. \tag{61}$$

Comparing (58) and (59) with (60) and (61), we conclude that

$$2I = s - 1 - (\alpha - 2)I,$$

which is equivalent to $I = (s-1)/\alpha$. We have therefore proved (56) for $\alpha \geq 2$.

Now I is an integer and so from (56), $s \equiv 1 \pmod{\alpha}$. If x_i is odd for all i then substitution of (56) in Lemma 5 gives

$$(s-1)/\alpha \equiv (s-1)/2 \pmod{2},$$

or equivalently $(s-1)(\alpha-2) \equiv 0 \pmod{4\alpha}$.

Finally apply Lemma 9 to show that (21) and (22) hold for $\alpha > 2$ and for all $1 \leq \beta \leq \alpha - 2$. (21) and (22) also hold for $\beta = 0$, since then (21) is vacuous and (22) is given by (55). Combining ranges, (21) and (22) hold for

$$\alpha \geq 2 \text{ and for all } 0 \leq \beta \leq \alpha - 2.$$

The substitution $r = \alpha - \beta$, together with (56), then shows that (53) and (54) hold for $\alpha \geq 2$ and for all $2 \leq r \leq \alpha$. $\square$

4 Nonexistence results for small α

In this section we use the results of Sections 2 and 3 to obtain nonexistence results for small values of α_j, where $t = \prod_j p_j^{\alpha_j}$ for distinct primes p_j. We express the nonexistence results in the form of restrictions on s and t.

In each case we state a theorem in terms of integers (x_i) and then a corollary in terms of an $s \times t$ binary array with Barker structure. Each corollary follows directly from the preceding theorem by letting (x_i) be the row sums of the array and using Lemma 1, as in the proof of Corollary 2.

We already know from Corollary 2 that $\alpha_j \geq 2$ for each j. The next case of interest is $\alpha_j = 2$ for all j. We first explore the case $\alpha = 2$ for some prime p.

Lemma 10 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying equations* (1)*, where $s > 1$ is odd, $x_i \neq 0$ for at least one odd i, and $k = 1$ or -1. Let p be a prime such that*

$$p^2 \,||\, t. \tag{62}$$

Then $p \,||\, x_0, x_{s-1}$ and

$$\begin{aligned} p &\;\mid\; x_i \text{ if and only if } i \neq (s-1)/2, \\ p^2 &\;\mid\; (x_j + x_{s-1-j}) \text{ for all } 0 \leq j < (s-1)/2. \end{aligned}$$

Proof By Lemma 8, $p \,||\, x_0, x_{s-1}$. Then by Theorem 2 (iii),

$$p \,|\, x_i \text{ if and only if } i \neq (s-1)/2. \tag{63}$$

We now show that

$$p^2 \,|\, (x_j + x_{s-1-j}) \text{ for all } 0 \le j < (s-1)/2. \tag{64}$$

For any $0 \le j < (s-1)/2$, put $u = (s-1)/2 - j$ in (1) and use (62) to show that

$$p^2 \,|\, \sum_i x_i x_{i+(s-1)/2-j}. \tag{65}$$

From (63), $p^2 \,|\, x_i x_{i+(s-1)/2-j}$ unless either $i = (s-1)/2$ or $i + (s-1)/2 - j = (s-1)/2$, so from (65), $p^2 \,|\, x_{(s-1)/2}(x_j + x_{s-1-j})$. By (63), $p \not| \, x_{(s-1)/2}$ and so $p^2 \,|\, (x_j + x_{s-1-j})$, proving (64). □

Subject to the condition $s > 3$, we now show that $\alpha_j > 2$ for some j and use Theorem 2 to restrict s when $\alpha_j = 3$ for some j. If $s = 3$, equations (1) have a solution in odd integers (x_i) with $k = -1$, namely $t = r^2$ for some odd r and $(x_0, x_1, x_2) = (r, \pm 1, -r)$.

Theorem 3 *Let $s, t, (x_i : 0 \le i < s)$ be integers satisfying equations* (1)*, where $s > 3$ and $t > 1$ are odd, $x_i \neq 0$ for all i, and $k = 1$ or -1. Then*

- *(i) there exists a prime p such that $p^3 \,|\, t$*
- *(ii) if $q^3 \,||\, t$ for some prime q and x_i is odd for all i then $s \equiv 1 \pmod{12}$*

Proof Since $t > 1$ we may write $t = \prod_j p_j^{\alpha_j}$, where the (p_j) are distinct primes and $\alpha_j \ge 1$ for all j. By Lemma 8, $\alpha_j \ge 2$ for all j. We seek a contradiction by supposing that $\alpha_j = 2$ for all j, so that

$$t = \prod_j p_j^2. \tag{66}$$

Applying Lemma 10,

$$p_j \quad || \quad x_0, x_{s-1} \text{ for all } j, \tag{67}$$

$$p_j \quad | \quad x_i \text{ if and only if } i \neq (s-1)/2, \text{ for all } j, \tag{68}$$

$$p_j^2 \quad | \quad (x_i + x_{s-1-i}) \text{ for all } 0 \le i < (s-1)/2, \text{ for all } j. \tag{69}$$

Using (66), we deduce from (68) and (69) that

$$\sqrt{t} \quad | \quad x_i \text{ for all } i \neq (s-1)/2, \tag{70}$$

$$t \quad | \quad (x_i + x_{s-1-i}) \text{ for all } 0 \le i < (s-1)/2. \tag{71}$$

Put $u = s - 1$ in (1) to obtain

$$x_0 x_{s-1} = \pm t. \tag{72}$$

Take $i = 0$, $s - 1$ in (70) and compare with (72) to show that

$$x_0 = \pm x_{s-1}. \tag{73}$$

For any j, take $i = 0$ in (69),

$$p_j^2 \,|\, (x_0 + x_{s-1}). \tag{74}$$

Suppose, if possible, that $x_0 = x_{s-1}$. Then from (74), $p_j^2 \,|\, 2x_0$ and so, since p_j is odd, $p_j^2 \,|\, x_0$. This contradicts (67) and so $x_0 \ne x_{s-1}$. From (73),

$$x_0 = -x_{s-1}. \tag{75}$$

Put $u = s - 2$ in (1) and substitute from (75),

$$x_0(x_{s-2} - x_1) = 0.$$

By hypothesis, $x_0 \ne 0$ and so

$$x_1 = x_{s-2}. \tag{76}$$

Take $i = 1$ in (71) and substitute from (76) to give $t \,|\, 2x_1$. Then since t is odd, $t \,|\, x_1$, and so from (76),

$$t \,|\, x_1, x_{s-2}. \tag{77}$$

We now force a contradiction by bounding $\sum_i x_i^2$ from below. By hypothesis, $1 < s - 2$ and so x_1, x_{s-2} are not the same variable. Therefore we may write

$$\sum_i x_i^2 = x_1^2 + x_{s-2}^2 + x_{(s-1)/2}^2 + \sum_{i \ne 1, s-2, (s-1)/2} x_i^2.$$

Since $x_i \neq 0$ for all i, from (70) and (77) we then have

$$\sum_i x_i^2 \geq t^2 + t^2 + 1 + (s-3)t$$

Comparing this bound with the value for the left side obtained by putting $u = 0$ in (1),

$$st + t - 1 \geq 2t^2 + 1 + (s-3)t,$$

which is equivalent to $(t-1)^2 \leq 0$. This contradicts $t > 1$ and so proves (i).

Suppose now that $q^3 \,||\, t$ for some prime q and x_i is odd for all i. From Lemma 8, either $q \,||\, x_0$ or $q \,||\, x_{s-1}$. We may therefore apply Theorem 2 (ii), reversing the order of the (x_i) if necessary, to show that $s - 1 \equiv 0 \pmod{12}$, proving (ii). □

Corollary 3 *Let A be an $s \times t$ binary array with Barker structure where $s > 3$ and $t > 1$ are odd. Then there exists a prime p such that $p^3 \,|\, t$. If $q^3 \,||\, t$ for some prime q then $s \equiv 1 \pmod{12}$.*

Given that $\alpha_j \geq 2$ for all j and $\alpha_k > 2$ for some k, we next consider the case $\alpha_k = 3$ for exactly one k and $\alpha_j = 2$ for all $j \neq k$.

Theorem 4 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying equations (1), where $s > 3$ and $t > 1$ are odd, x_i is odd for all i, and $k = 1$ or -1. Let $t = q^3 \prod_j p_j^{\alpha_j}$, where $q, (p_j)$ are distinct primes and $\alpha_j \geq 1$ for all j. Then $\alpha_j > 2$ for some j.*

Proof By Lemma 8, $\alpha_j \geq 2$ for all j. Suppose, for a contradiction, that $\alpha_j = 2$ for all j, so that

$$t = q^3 \prod_j p_j^2. \tag{78}$$

By Lemma 10,

$$p_j \,|\, x_i \text{ for all } i \neq (s-1)/2, \text{ for all } j. \tag{79}$$

By Lemma 8, either $q \,||\, x_0$ or $q \,||\, x_{s-1}$. We may assume, by reversing the order of the x_i if necessary, that $q \,||\, x_0$. Then by Theorem 2 (iii) and (iv),

$$\begin{aligned} q^2 &\;|\; x_i \text{ for all } 2(s-1)/3 < i \leq s-1, \\ q &\;|\; x_i \text{ for all } 0 \leq i \leq 2(s-1)/3,\; i \neq (s-1)/3. \end{aligned}$$

Together with (79), this implies that

$$
\begin{aligned}
q^2\prod_j p_j &\mid x_i \text{ for all } 2(s-1)/3 < i \le s-1, \\
q\prod_j p_j &\mid x_i \text{ for all } 0 \le i \le 2(s-1)/3,\ i \ne (s-1)/3, (s-1)/2, \\
\prod_j p_j &\mid x_{(s-1)/3}.
\end{aligned}
$$

Since $x_i \neq 0$ for all i, we can therefore bound $\sum_i x_i^2$ from below,

$$\sum_i x_i^2 \ge \frac{(s-1)q^4}{3}\prod_j p_j^2 + \left(\frac{2(s-1)}{3} - 2\right) q^2 \prod_j p_j^2 + \prod_j p_j^2.$$

Comparing this bound with the value for the left hand side obtained by putting $u = 0$ in (1), and making the substitution $\prod_j p_j^2 = t/q^3$ from (78),

$$s+1 \ge \frac{(s-1)q}{3} + \frac{2s-8}{3q} + \frac{1}{q^3}.$$

Rearrangement gives

$$s \le \frac{q^4+3q^3+8q^2-3}{q^2(q-1)(q-2)},$$

which can be written as

$$s \le 1 + 3f(q) \tag{80}$$

where

$$f(q) = \frac{2q^3+2q^2-1}{q^2(q-1)(q-2)}.$$

It is easy to check that

$$
\begin{aligned}
f(q) - f(q+1) &= \frac{2q^4+12q^3+18q^2+4q-1}{(q+1)^2q^2(q-1)(q-2)} \\
&> 0 \text{ for all } q \ge 3. \qquad (81)
\end{aligned}
$$

Now q is an odd prime and so $q \ge 3$. Therefore, from (80) and (81),

$$s \le 1 + 3f(3) = 77/6 < 13. \tag{82}$$

But by Theorem 3 (*ii*), $s \equiv 1 \pmod{12}$, and by hypothesis $s > 3$. This contradicts (82), completing the proof. □

Corollary 4 *Let A be an $s \times t$ binary array with Barker structure where $s > 3$ and $t > 1$ are odd. Let $t = q^3 \prod_j p_j^{\alpha_j}$, where $q, (p_j)$ are distinct primes and $\alpha_j \geq 1$ for all j. Then $\alpha_j > 2$ for some j.*

The final case we shall consider is $\alpha_j = 2$ or 4 for all j. We first explore the case $\alpha = 4$ for some prime p. By Lemma 8, $p^\gamma \,||\, x_0$ where $\gamma = 1$, 2 or 3. The values $\gamma = 1$ or 3 are covered by Theorem 2, leaving only the value $\gamma = 2$ to deal with.

Lemma 11 *Let $s, (x_i : 0 \leq i < s)$ be integers and let p be an odd prime such that*

$$p^4 \quad | \quad \sum_i x_i x_{i+u} \text{ for all } 0 < u < s, \tag{83}$$

$$p^2 \quad || \quad x_0, \tag{84}$$

$$p^2 \quad || \quad x_{s-1}. \tag{85}$$

Let $0 \leq I < s$ be an integer such that

$$p \,|\, x_i \text{ if and only if } i \neq I. \tag{86}$$

Then

$$I \quad = \quad (s-1)/2, \tag{87}$$

$$p^2 \quad | \quad x_j, x_{s-1-j} \text{ for all } 0 \leq j \leq \lfloor (s-3)/4 \rfloor. \tag{88}$$

If also

$$x_0 = -x_{s-1} \tag{89}$$

then

$$p^2 \,|\, x_j \text{ for all } j \neq (s-1)/2. \tag{90}$$

Proof We may assume, by reversing the order of the (x_i) if necessary, that

$$I \leq (s-1)/2. \tag{91}$$

We show, by induction on j, that

$$p^2 \,|\, x_j, x_{s-1-j} \text{ for all } 0 \leq j \leq \lfloor (I-1)/2 \rfloor. \tag{92}$$

The case $j = 0$ is given by (84) and (85). Assume that for some

$$1 \le j \le \lfloor (I-1)/2 \rfloor, \tag{93}$$

$$p^2 \,|\, x_k, x_{s-1-k} \text{ for all } 0 \le k < j. \tag{94}$$

Put $u = s - 1 - 2j$ in (83), showing that

$$p^3 \,|\, (\sum_{i=0}^{j-1} x_i x_{i+s-1-2j} + x_j x_{s-1-j} + \sum_{i=j+1}^{2j} x_i x_{i+s-1-2j}) \tag{95}$$

By (94), $p^2 \,|\, x_i$ for all $0 \le i \le j-1$. By (91) and (93), $s - 1 - 2j > I$ and so by (86), $p \,|\, x_{i+s-1-2j}$ for all $0 \le i \le j-1$. Therefore $p^3 \,|\, \sum_{i=0}^{j-1} x_i x_{i+s-1-2j}$. Similarly $p^3 \,|\, \sum_{i=j+1}^{2j} x_i x_{i+s-1-2j}$. Then from (95),

$$p^3 \,|\, x_j x_{s-1-j}$$

and so

$$\text{either } p^2 \,|\, x_j \text{ or } p^2 \,|\, x_{s-1-j}. \tag{96}$$

Now take $u = s - 1 - j$ in (83),

$$p^4 \,|\, (x_0 x_{s-1-j} + \sum_{i=1}^{j-1} x_i x_{i+s-1-j} + x_j x_{s-1}). \tag{97}$$

By (94), $p^2 \,|\, x_i, x_{i+s-1-j}$ for all $1 \le i \le j-1$ and so $p^4 \,|\, \sum_{i=1}^{j-1} x_i x_{i+s-1-j}$. Therefore from (97),

$$p^4 \,|\, (x_0 x_{s-1-j} + x_j x_{s-1}).$$

Then from (84) and (85),

$$p^2 \,|\, x_j \text{ if and only if } p^2 \,|\, x_{s-1-j}.$$

Therefore, using (96),

$$p^2 \,|\, x_j, x_{s-1-j},$$

completing the induction on j and proving (92).

Put $u = s - 1 - I$ in (83) to show that

$$p^3 \,|\, (\sum_{i=0}^{I-1} x_i x_{i+s-1-I} + x_I x_{s-1}). \tag{98}$$

We next prove (87), considering separately the cases I even and I odd.

Suppose firstly that I is odd, so that (92) and (98) become

$$p^2 \quad | \quad x_j, x_{s-1-j} \text{ for all } 0 \leq j \leq (I-1)/2, \tag{99}$$

$$p^3 \quad | \quad (\sum_{i=0}^{(I-1)/2} x_i x_{i+s-1-I} + \sum_{i=(I+1)/2}^{I-1} x_i x_{i+s-1-I} + x_I x_{s-1}). \tag{100}$$

From (99), $p^2 \,|\, x_{i+s-1-I}$ for all $(I+1)/2 \leq i \leq I-1$ and so by (86), $p^3 \mid \sum_{i=(I+1)/2}^{I-1} x_i x_{i+s-1-I}$. Therefore from (100),

$$p^3 \,|\, (\sum_{i=0}^{(I-1)/2} x_i x_{i+s-1-I} + x_I x_{s-1}). \tag{101}$$

From (86), $p \nmid x_I$ and so by (85), $p^2 \,||\, x_I x_{s-1}$. Therefore from (101),

$$p^2 \,||\, \sum_{i=0}^{(I-1)/2} x_i x_{i+s-1-I}. \tag{102}$$

Now from (99), $p^2 \,|\, x_i$ for all $0 \leq i \leq (I-1)/2$. Suppose, if possible, that $s-1-I > I$. Then by (86), $p \,|\, x_{i+s-1-I}$ for all $0 \leq i \leq (I-1)/2$ and so $p^3 \mid \sum_{i=0}^{(I-1)/2} x_i x_{i+s-1-I}$, contradicting (102). Therefore $s-1-I \leq I$, which combines with (91) to give $I = (s-1)/2$.

Suppose instead that I is even, so that (92) and (98) become

$$p^2 \quad | \quad x_j, x_{s-1-j} \text{ for all } 0 \leq j \leq I/2-1, \tag{103}$$

$$p^3 \quad | \quad (\sum_{i=0}^{I/2-1} x_i x_{i+s-1-I} + x_{I/2} x_{s-1-I/2} + \sum_{i=I/2+1}^{I-1} x_i x_{i+s-1-I} + x_I x_{s-1}). \tag{104}$$

Suppose, if possible, that

$$s - 1 - I > I. \tag{105}$$

From (103), $p^2 \,|\, x_i$ for all $0 \leq i \leq I/2-1$ and $p^2 \,|\, x_{i+s-1-I}$ for all $I/2+1 \leq i \leq I-1$. Hence by (86) and (105), $p^3 \,|\, (\sum_{i=0}^{I/2-1} x_i x_{i+s-1-I} + \sum_{i=I/2+1}^{I-1} x_i x_{i+s-1-I})$, and so from (104),

$$p^3 \,|\, (x_{I/2} x_{s-1-I/2} + x_I x_{s-1}).$$

As before, $p^2 \,||\, x_I x_{s-1}$ and therefore

$$p^2 \,||\, x_{I/2} x_{s-1-I/2}.$$

It follows from (86) and (91) that

$$p \quad || \quad x_{I/2}, \tag{106}$$

$$p \quad || \quad x_{s-1-I/2}. \tag{107}$$

Apply Lemma 7 (*ii*) for all $0 \le j < I/2$ so that from (103), $p^2 \,|\, x_j$ for all $3I/2 < j \le 2I$. Apply Lemma 6 to show that $p^2 \,|\, x_j$ for all $2I < j < s$. Combine to give

$$p^2 \,|\, x_j \text{ for all } 3I/2 < j < s. \tag{108}$$

Apply Lemma 7 (*i*) with $j = I/2$ so that from (106),

$$p \,||\, x_{3I/2}. \tag{109}$$

Comparing (103) and (107) with (108) and (109), we conclude that

$$s - 1 - I/2 = 3I/2,$$

contradicting (105). Therefore $s - 1 - I \le I$, which combines with (91) to give (87).

We therefore have $I = (s-1)/2$ regardless of whether I is even or odd, and the form (88) is obtained by substituting for I in (92).

Suppose finally that (89) holds. By (87), the form (90) is equivalent to

$$p^2 \,|\, x_j, x_{s-1-j} \text{ for all } 0 \le j < I, \tag{110}$$

which we prove by induction on j. The case $j = 0$ is given by (88). Assume that for some

$$0 < j < I, \tag{111}$$

$$p^2 \,|\, x_k, x_{s-1-k} \text{ for all } 0 \le k < j. \tag{112}$$

Put $u = s - 1 - j$ in (83),

$$p^4 \,|\, (x_0 x_{s-1-j} + \sum_{i=1}^{j-1} x_i x_{i+s-1-j} + x_j x_{s-1}). \tag{113}$$

By (112), $p^4 \mid \sum_{i=1}^{j-1} x_i x_{i+s-1-j}$ and then substitution from (89) in (113) gives

$$p^4 \,|\, x_{s-1}(x_j - x_{s-1-j}).$$

Then from (85),

$$p^2 \,|\, (x_j - x_{s-1-j}). \tag{114}$$

By (111), $I - j > 0$ so we may take $u = I - j$ in (83) and use (87) to show that

$$p^2 \,|\, (\sum_{i \neq j, I} x_i x_{i+I-j} + x_j x_I + x_I x_{s-1-j}).$$

From (86), $p^2 \mid \sum_{i \neq j, I} x_i x_{i+I-j}$ and so

$$p^2 \,|\, x_I(x_j + x_{s-1-j}).$$

But $p \not| \, x_I$ by (86), and therefore

$$p^2 \,|\, (x_j + x_{s-1-j}). \tag{115}$$

Summing (114) and (115), $p^2 \,|\, 2x_j$ and, since p is odd, $p^2 \,|\, x_j$. Therefore from (115),

$$p^2 \,|\, x_j, x_{s-1-j},$$

completing the induction on j and proving (110) and therefore (90). □

We can now treat the case $\alpha_j = 2$ or 4 for all j.

Theorem 5 *Let $s, t, (x_i : 0 \leq i < s)$ be integers satisfying equations* (1)*, where $s > 3$ and $t > 1$ are odd, $x_i \neq 0$ for all i, and $k = 1$ or -1. Let*

$$t = \left(\prod_j p_j^2 \right) \left(\prod_k q_k^4 \right), \tag{116}$$

where the (p_j, q_k) are distinct primes. Then $s \equiv 1 \pmod 4$.

Proof Suppose, for a contradiction, that

$$s \equiv 3 \pmod 4. \tag{117}$$

Applying Lemma 10,

$$p_j \quad || \quad x_0, x_{s-1} \text{ for all } j, \tag{118}$$

$$p_j \quad | \quad x_i \text{ if and only if } i \neq (s-1)/2, \text{ for all } j. \tag{119}$$

By Lemma 8,

$$q_k^{\gamma_k} \,||\, x_0, \quad q_k^{4-\gamma_k} \,||\, x_{s-1} \text{ for all } k, \tag{120}$$

where each $\gamma_k = 1$, 2 or 3. By Theorem 2 (i), if $\gamma_k = 1$ or 3 for any k then $s \equiv 1 \pmod 4$, contradicting (117), and so from (120),

$$q_k^2 \,||\, x_0, x_{s-1} \text{ for all } k. \tag{121}$$

Then by Lemmas 3 and 11,

$$q_k \,|\, x_i \text{ if and only if } i \neq (s-1)/2, \text{ for all } k. \tag{122}$$

Using (116), we deduce from (118) and (121) that

$$\sqrt{t} \,|\, x_0, x_{s-1}. \tag{123}$$

Put $u = s-1$ in (1), giving $x_0 x_{s-1} = \pm t$. Then (123) implies that

$$x_0 = \pm x_{s-1}. \tag{124}$$

Now from (116) and Theorem 3 (i), there exists some k such that $q_k^4 \,|\, t$. For any such k, take $u = (s-1)/2$ in (1) and use (117) to write

$$q_k^3 \,|\, (x_0 x_{(s-1)/2} + \sum_{i=1}^{(s-3)/4} x_i x_{i+(s-1)/2} + \sum_{i=(s+1)/4}^{(s-3)/2} x_i x_{i+(s-1)/2} + x_{(s-1)/2} x_{s-1}). \tag{125}$$

Applying Lemma 11,

$$q_k^2 \,|\, x_i, x_{s-1-i} \text{ for all } 0 \leq i \leq (s-3)/4,$$

which, together with (122), implies that $q_k^3 \,|\, (\sum_{i=1}^{(s-3)/4} x_i x_{i+(s-1)/2} + \sum_{i=(s+1)/4}^{(s-3)/2} x_i x_{i+(s-1)/2})$. Therefore from (125),

$$q_k^3 \,|\, x_{(s-1)/2}(x_0 + x_{s-1}),$$

and since, by (122), $q_k \nmid x_{(s-1)/2}$,

$$q_k^3 \,|\, (x_0 + x_{s-1}). \tag{126}$$

Suppose, if possible, that $x_0 = x_{s-1}$. Then from (126), $q_k^3 \,|\, 2x_0$ and so, since q_k is odd, $q_k^3 \,|\, x_0$. This contradicts (121) and so $x_0 \neq x_{s-1}$. From (124),

$$x_0 = -x_{s-1}. \tag{127}$$

Now we can apply Lemma 11 to obtain

$$q_k^2 \,|\, x_i \text{ for all } i \neq (s-1)/2, \text{ for all } k. \tag{128}$$

Together with (116) and (119), this gives

$$\sqrt{t} \,|\, x_i \text{ for all } i \neq (s-1)/2. \tag{129}$$

Take $u = s - 2$ in (1) and substitute from (127),

$$x_0(x_{s-2} - x_1) = 0.$$

Since $x_0 \neq 0$,

$$x_1 = x_{s-2}. \tag{130}$$

Next take $u = (s-3)/2$ in (1),

$$t \,|\, (x_1 x_{(s-1)/2} + \sum_{i \neq 1, (s-1)/2} x_i x_{i+(s-3)/2} + x_{(s-1)/2} x_{s-2}). \tag{131}$$

Now from (116), $p_j^2 \,|\, t$ for all j and so (119) and (131) imply that $p_j^2 \,|\, (x_1 + x_{s-2})$ for all j. Then (130) gives $p_j^2 \,|\, x_1, x_{s-2}$ for all j. Similarly $q_k^4 \,|\, t$ for all k and so (128), (130) and (131) imply that $q_k^4 \,|\, x_1, x_{s-2}$. Combining and using (116),

$$t \,|\, x_1, x_{s-2}. \tag{132}$$

We now proceed as in the proof of Theorem 3 (i), using (129) and (132) to show that $(t-1)^2 \leq 0$, contradicting $t > 1$. Therefore we conclude that (117) is false and hence $s \equiv 1 \pmod 4$. $\square$

Corollary 5 *Let A be an $s \times t$ binary array with Barker structure where $s > 3$ and $t > 1$ are odd. Let $t = \prod_j p_j^{\alpha_j}$, where the (p_j) are distinct primes and $\alpha_j = 2$ or 4 for all j. Then $st \equiv 1 \pmod 4$.*

Proof By Theorem 5, $s \equiv 1 \pmod 4$. Since t is the product of even powers of primes, $t \equiv 1 \pmod 4$. Therefore $st \equiv 1 \pmod 4$. □

This completes our analysis for small values of α_j.

The nonexistence results in this paper, for $s \times t$ binary arrays with Barker structure where s, t are odd, are all based on equations (1). Using equations (2) as well as (1) we may interchange s and t in each of our results. In particular we can exclude the case $s = 3$, $t > 1$ by Corollary 2. We conclude this section by summarising the nonexistence results arising from both (1) and (2), although for clarity we mostly do not repeat results with s and t interchanged.

Theorem 6 *Let $A = (a_{ij})$ be an $s \times t$ binary array with Barker structure where s, t are odd and $s > 1$. If $st \equiv 1 \pmod 4$ then $2st - 1 = (\sum_i \sum_j a_{ij})^2$, $s \equiv t \equiv 1 \pmod 4$ and $p \equiv 1 \pmod 4$ for each prime p dividing s or t. If $t = 1$ then $s = 3$, 5, 7, 11 or 13. Otherwise, if $t > 1$, write $t = \prod_j p_j^{\alpha_j}$ where the (p_j) are distinct primes and $\alpha_j \geq 1$ for all j. Then*

(i) $\alpha_j \geq 2$ *for all* j

(ii) $\alpha_k > 2$ *for some* k

(iii) *if* $\alpha_k = 3$ *for some* k *then* $s \equiv 1 \pmod{12}$

(iv) *if* $\alpha_k = 3$ *for some* k *then* $\alpha_j > 2$ *for some* $j \neq k$

(v) *if* $\alpha_j = 2$ *or* 4 *for all* j *then* $st \equiv 1 \pmod 4$.

5 Comments

The smallest odd value of $st > 13$ for which the nonexistence of an $s \times t$ binary array with Barker structure is not determined by Theorem 6 occurs at $\{s, t\} = \{3^5, 3^6\}$. The existence of such an array implies the existence of a (177147, 88573, 44286)-difference set in $\mathbb{Z}_{243} \times \mathbb{Z}_{729}$ [2].

In our opinion, the apparent scarcity of solutions to the necessary equations, both in the row and column sums, provides good reason to doubt the existence of an $s \times t$ binary array with Barker structure where $st > 13$ is odd.

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